# Numerical Analysis of the Asymptotic Two-Point Boundary Value Solution for $N$-Body Trajectories

J. E. LANCASTER* AND R. A. ALLEMANN†
*McDonnell Douglas Astronautics Company, Huntington Beach, Calif.*

## Theme

SECOND-ORDER asymptotic boundary value solutions to the problem of $N$-bodies are summarized. The solutions have been formulated to solve the boundary value problem without iterations. Analytical expressions in the form of asymptotic expansions give all the unknown parameters as functions of the prescribed boundary conditions. Computation times are comparable to the standard conic approximations with considerable increase in accuracy.

## Contents

The method of matched asymptotic expansions was first applied to the problem of three bodies by Lagerstrom and Kevorkian[1] and later by Breakwell and Perko.[2] Their initial value solutions were extended to boundary value problems by Lancaster[3] and Carlson.[4] A recent study by Lancaster[5] combined the best features of the earlier work, extended the theory to second order, and derived a number of two-point boundary value solutions.

The problem of $N$-bodies for which an asymptotic solution is desired consists of finding the motion of a body of negligible mass (referred to as the particle) under the influence of one primary body and $N$-2 secondary bodies whose motions relative to the primary body are known. The $N$-body differential equation can be written as

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3} - \sum_{i=1}^{N-2} \mu_i \left[ \frac{\mathbf{r}-\mathbf{p}_i}{|\mathbf{r}-\mathbf{p}_i|^3} + \frac{\mathbf{p}_i}{p_i^3} \right] \tag{1}$$

where $r$ is the position of the particle with respect to the primary body, $p_i$ is the position of the $i$th secondary body with respect to the primary body, and $\mu_i$ is the dimensionless mass of the $i$th secondary body.

The basic mathematical tool used to obtain an analytical solution of Eq. (1) is the second-order asymptotic expansion of the form

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \mu\mathbf{r}_1(t) + \mu^2\mathbf{r}_2(t) + O(\mu^3) \tag{2}$$

where $\mu$ is one of the $\mu_i$. This expansion can be substituted directly into Eq. (1) and solutions found for $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$. The expansion is valid as long as $\mathbf{r}-\mathbf{p}_i = O(1)$ in Eq. (1) and is termed the outer expansion (or solution). The leading term, $\mathbf{r}_0(t)$, is simply the two-body solution which results in elliptic motion about the primary. The higher-order terms are found as definite integrals which must be evaluated by Gaussian quadrature or a similar numerical technique.

If for some $k$, $\mathbf{r}-\mathbf{p}_k = O(\mu_k)$, then Eq. (2) contains functions which are singular in the limit as $\mathbf{r}-\mathbf{p}_k \to 0$. In this limit, another solution, termed the inner solution, can be obtained from Eq. (1). Introducing the change of variables

$$\mathbf{R}_k = (\mathbf{r}-\mathbf{p}_k)/\mu_k \tag{3}$$

$$S_k = (t-t_{pk})/\mu_k \tag{4}$$

transforms Eq. (1) into

$$d^2\mathbf{R}_k/dS_k^{\,2} = -\mathbf{R}_k/R_k^{\,3} + \mathbf{P}(\mathbf{R}_k, \mathbf{p}_i) \tag{5}$$

where $t_{pk}$ is the time of closest approach to the $k$th body and the forcing function $\mathbf{P}$ is order $\mu_k^{\,2}$. The inner expansion takes the form

$$\mathbf{R}_k(S_k) = \mathbf{R}_{k0}(S_k) + \mu_k^{\,2}\mathbf{R}_{k2}(S_k) + O(\mu_k^{\,3}) \tag{6}$$

where $R_{k0}$ is the two-body hyperbola about the $k$th body and $\mathbf{R}_{k2}$ is a definite integral.

For a trajectory making a close approach to the $k$th body, there is a continuous transition from the outer to the inner solution. In the transition region, or overlap domain, the two solutions are equivalent; i.e., they have identical singularities which can be removed through a limit process called matching. The matching results in relationships between the constants of the outer and inner solutions which form the basis for the derivation of several different boundary value solutions. Reference 5 discusses solutions of the following type: Earth-to-moon, interplanetary, midcourse (both lunar and interplanetary), and one- and two-impulse moon-to-Earth.

To illustrate the nature of the asymptotic solutions the Earth-to-moon boundary value problem is shown in Fig. 1. The initial time $t_0$, the initial position relative to the Earth, $\mathbf{r}(t_0)$, and the pericynthion radius, inclination, and time are all prescribed. The initial velocity relative to the Earth is unknown and must be determined from the asymptotic solution.

The zero-order solution, shown as the dashed line in Fig. 1, is the solution of a Lambert problem between the initial position, $\mathbf{r}(t_0)$, and the final position, $\mathbf{p}_M(t_M)$, which is the position of the moon at the specified time of pericynthion passage. The asymptotic boundary value solution is then used to determine first- and second-order corrections to the Lambert velocity at $t_0$. The end result is the solution shown as $\mathbf{r}(t)$ which satisfies the prescribed terminal conditions at pericynthion.

Trajectories of the type shown in Fig. 1 were compared with numerical integration. Initial radius, true anomaly, and flight time are shown in Table 1. The prescribed pericynthion conditions were 1200-naut-mile radius ($\bar{\rho}_M$) and $-35°$ arrival inclination ($\bar{\imath}_M$; the negative sign indicating approach from under the moon).

Cases 102 and 111–117 show the effect of moving the initial position away from the Earth. For the first-order solution, the error in time-of-flight is nearly constant. Errors in pericynthion radius and inclination show that the first-order solution is getting better as the initial position moves out until a point around 150,000 naut miles from Earth is reached. Case 117 then shows a slight increase in the radius error.

---

Presented as Paper 72-49 at the AIAA 10th Aerospace Sciences Meeting, San Diego, Calif., January 17–19, 1972; submitted March 1, 1972; synoptic received September 21, 1972. Full paper available from AIAA. Price: Microfiche, $1.00; hard copy, $5.00. **Order must be accompanied by remittance.** The work reported was sponsored by McDonnell Douglas Astronautics Company under Independent Research and Development and by NASA Contract NAS9-10526.

Index category: Lunar and Planetary Trajectories.

* Supervisor, Guidance and Flight Mechanics Department. Member AIAA.

† Engineer Scientist, Guidance and Flight Mechanics Department. Associate Member AIAA.

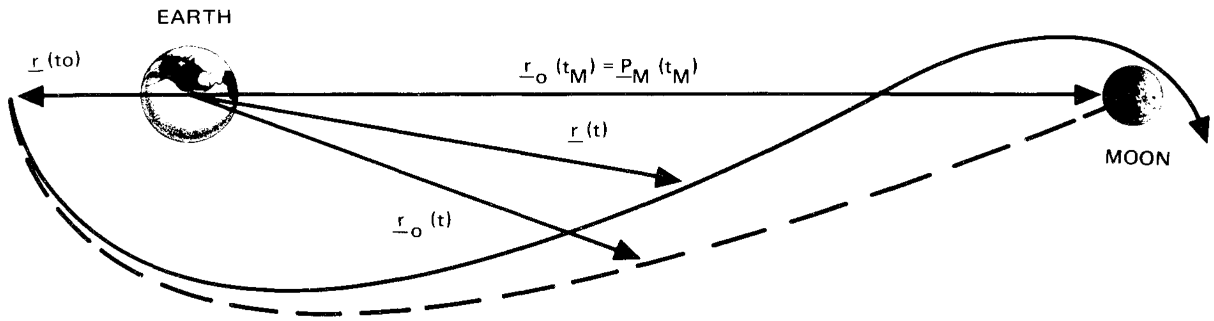


**Fig. 1 Earth-to-moon solution. (Letters with bars under are boldface in the text.)**

The second-order solution shows a slightly more rapid increase in the time-of-flight error as the initial position moves away from the Earth. The pericynthion radius error, however, is initially reduced, dropping from 299 nautical miles for Case 102 down to 2 naut miles for Case 113. It then increases, becoming larger than first-order.

Similar comparisons were made for the other types of boundary value solutions. The best results were obtained for the interplanetary midcourse solution where second-order errors in pericenter time, radius, and inclination were as small as $10^{-1}$ sec, $10^{-1}$ naut mile, and $10^{-3}$°, respectively, starting from midcourse points along a reference 244-day Earth-to-Mars transfer.

The full interplanetary solution (originating close to one planet, terminating close to another) and the two-impulse moon-to-Earth solution resulted in errors somewhat larger than anticipated. In several cases the second-order errors were larger than the corresponding first-order errors (as in Cases 102, 116, and 117 in Table 1).

FORTRAN programs used to evaluate the asymptotic solutions were run on a CDC 6500 computer as was the numerical integration program. The computation time varied from 1.7 sec for a moon-to-Earth trajectory ($N = 4$) up to 6.0 sec for a complete Earth-to-Mars trajectory ($N = 7$).

The accuracy of the asymptotic solution requires some discussion. It is apparent from Eq. (2) that smaller values of $\mu$ should give better results. This was verified by comparing lunar and interplanetary results. It was also found, however, that for a fixed value of $\mu$ the accuracy of the asymptotic solution (particularly the second-order) is also dependent on the boundary conditions of the specific problem. Not only do these boundary conditions affect the magnitude of the errors but cases arise where the second-order error is larger than first-order. An explanation of this divergence requires an examination of all of the data generated for Ref. 5. It was found that the cases with large second-order errors were cases where the first-order correction to the Lambert velocity at $t_0$ was quite large. Thus, in Fig. 1, the difference between the Lambert solution (dashed line) and the first-order solution was large, i.e., the function $r_1(t)$, which is initially zero, grew rapidly with $t$. In the second-order solution the effect of $r_1(t)$ enters quadratically and when $r_1(t)$ is large the integration of the quadratic term over the entire trajectory results in a large second-order correction. This breaks down the assumption of uniformity in Eq. (2) and leads to excessive second-order errors. Such a result should not be totally unexpected since it is known that asymptotic expansions which are initially convergent may diverge after $n$ terms. The results obtained thus far indicate that the derived form of the asymptotic $N$-body solution may, for certain choices of the boundary conditions, begin to diverge after two terms.

The rapid computational speed of the asymptotic solution makes it attractive for applications where a more accurate approximation than that offered by conic approximations is desired. The results thus far indicate that determination of midcourse velocity corrections is the best application. In addition, interplanetary applications result in significantly more accurate solutions than lunar applications.

**Table 1 Boundary conditions and accuracies for Earth-to-moon trajectories**

| Case no. | $r(t_0)$, naut miles | $f(t_0)$, deg | $t_{pM}$, hr | Order of solution | $\Delta t_{pM}$, min | $\Delta \bar{\rho}_M$, naut miles | $\Delta \bar{\imath}_M$, deg |
|---|---|---|---|---|---|---|---|
| 102 | 3,544 | 10 | 80 | 1 | 36 | −92 | −0.05 |
| | | | | 2 | −8 | −299 | −0.33 |
| 111 | 12,850 | 118 | 79 | 1 | 36 | −82 | −0.18 |
| | | | | 2 | 8 | −70 | 0.19 |
| 112 | 40,162 | 149 | 75 | 1 | 36 | −71 | −0.16 |
| | | | | 2 | 13 | −9 | 0.14 |
| 113 | 63,707 | 157 | 70 | 1 | 37 | −62 | −0.14 |
| | | | | 2 | 15 | −2 | 0.14 |
| 114 | 98,489 | 164 | 60 | 1 | 38 | −50 | −0.10 |
| | | | | 2 | 16 | −12 | 0.12 |
| 115 | 125,011 | 167 | 50 | 1 | 38 | −44 | −0.07 |
| | | | | 2 | 17 | −32 | 0.08 |
| 116 | 146,533 | 170 | 40 | 1 | 39 | −40 | −0.04 |
| | | | | 2 | 19 | −54 | 0.06 |
| 117 | 164,463 | 172 | 30 | 1 | 40 | −42 | −0.02 |
| | | | | 2 | 20 | −83 | 0.04 |

## References

[1] Lagerstrom, P. A. and Kevorkian, J., "Earth-to-Moon Trajectories in the Restricted Three-Body Problem," *Journal de Mecanique*, Vol. 2, No. 2, June 1963, pp. 189–218.

[2] Breakwell, J. V. and Perko, L. M., "Matched Asymptotic Expansions, Patched Conics, and the Computation of Interplanetary Trajectories," *AIAA Progress in Astronautics and Aeronautics: Methods in Astrodynamics and Celestial Mechanics*, Vol. 17, edited by R. L. Duncombe and V. G. Szebehely, Academic Press, New York, 1966, pp. 159–182.

[3] Lancaster, J. E., "Numerical Analysis of the Asymptotic Two-Point Boundary Value Solution for Moon-to-Earth Trajectories," AIAA Paper 70-1060, Santa Barbara, Calif., 1970.

[4] Carlson, N. A., "An Explicit Analytic Guidance Formulation for Many-Body Space Trajectories," Ph.D. dissertation, May 1969. Dept. of Aeronautics and Astronautics, MIT, Cambridge, Mass.; also AIAA Paper 71-117, New York, 1971.

[5] Lancaster, J. E., "Application of Matched Asymptotic Expansions to Lunar and Interplanetary Trajectories," MDC G2748, Feb. 1972, Vols. 1 and 2, McDonnell Douglas Astronautics Co., Huntington Beach, Calif.; also NASA CR, 1973.